Nonlinear Motion of a Missile with Slight Configurational Asymmetries

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The quasi-linear analysis which has been successfully applied to the nonlinear angular motion of symmetric missiles is extended to include the nonlinear motion of slightly asymmetric missiles. This analysis indicates the existence of three special spin rates—zero spin, resonant spin, and approximately three times resonant spin. The exact solution for the small amplitude motion about a large trim angle of a nonspinning missile with a cubic static moment is obtained and the quasi-linear relations are shown to agree well with this exact result. Relations for a cubic damping moment and a cubic lift force are derived.

Nomenclature

A = $(\rho Sl^3/2I_y)(C_{m_0} + iC_{n_0})$
c_0,c_2 = cubic static moment coefficients
c_2^* = nonlinear damping moment coefficient, Eq. (39)
C_{L_α} = lift coefficient
$C_{\tilde{m}},C_{\tilde{n}}$ = coefficients of the transverse components of the aerodynamic moment
C_{m_0},C_{n_0} = aerodynamic moment coefficients due to asymmetry
C_{Y_0},C_{Z_0} = aerodynamic force coefficients due to asymmetry
$C_{M_{\dot{\alpha}}},C_{M_q}$ = damping moment coefficients, Eq. (1)
d_0,d_2 = damping moment coefficients, Eq. (39)
f,g = see Eq. (42)
f_0,f_2 = cubic lift force coefficients, Eq. (44)
f_R = range value of the lift coefficient
H = $-(\rho Sl^3/2I_y)(C_{M_q} + C_{M_{\dot{\alpha}}})$
H_0,H_2 = coefficients in $H = H_0 + H_2\delta^2$
I_x,I_y = axial, transverse moments of inertia
K_j = amplitude of j-mode, $(j = 1,2)$
K_3 = amplitude of the response to asymmetric moment
l = reference length
L = distance used for averages over flight path
m = mass
m_3 = $M_2K_3^2/M_0$
M_i = $(\rho Sl^3/2I_y)c_i$
p = $d\phi/dt$, roll rate
P = $(I_x/I_y)(pl/V)$, gyroscopic spin
$\tilde{q},\tilde{r}$ = angular velocity components along $\tilde{Y},\tilde{Z}$ axis
s = dimensionless distance along flight path $\int_{t_0}^{t}\left(\frac{V}{l}\right)dt$
S = reference area
$\tilde{v},\tilde{w}$ = $\tilde{Y},\tilde{Z}$ components of velocity
V = magnitude of velocity
x_2,x_3 = components of swerving motion, Eq. (43)
$X,\tilde{Y},\tilde{Z}$ = nonrolling Cartesian coordinate axes
δ = $|\xi|$ sine of total angle of attack
δ_{ej}^2 = effective squared yaws, Eq. (15)
δ_e^2 = effective squared yaw, Eq. (16)
$(\delta_{ej}^2)_r$ = effective squared yaws of resonance, Eqs. (20) and (24)
δ_{es}^2 = effective squared yaw for swerve, Eq. (45)
λ_j = damping rate of the j modal amplitude, K_j'/K_j
λ_j^* = damping rate for constant frequency
$\tilde{\mu}$ = $(\tilde{q} + i\tilde{r})l/V$
ξ = $(\tilde{v} + i\tilde{w})/V$
ρ = air density
ϕ = roll angle
ϕ_j = j modal phase angle, $\phi_{j0} + \phi_j's$ $(j = 1,2)$
ϕ_3 = response phase angle, $\phi_{30} + \phi$
$\hat{\phi},\phi_r$ = $\phi_1 - \phi_2$, and $\phi_1 - \phi_3$, respectively
ϕ^*,ϕ^{**} = $(\hat{\phi} - 2\phi_r)/2$, and $(\hat{\phi} + \phi_r)/2$, respectively

Superscripts

$-$ = complex conjugate
$\sim$ = component in a nonrolling coordinate system
$(\ )'$ = derivatives with respect to arclength, s

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Introduction

IN 1956 an analysis technique[1] was developed to obtain cubic moment and force coefficients from ballistic range tests of symmetric missiles. As the result of the great success of this approach it was extended to the prediction of symmetric missile motion influenced by quite general nonlinear forces and moments.[2–6] Almost all of this work, however, was limited to rotationally symmetric missiles.

The linear motion of "slightly asymmetric" missile has been developed in some detail by Nicolaides.[7] The aerodynamic force and moment expansion for a slightly asymmetric missile has the same form as that for a symmetric missile with the exception of the presence of constant amplitude force and moment terms whose orientations are fixed relative to the missile. These terms induce a trim angle of attack which rolls with the missile (tricyclic motion). The amplitude of this angle of attack varies with the roll rate and reaches a maximum when the roll rate equals the natural pitch frequency (resonance). Later Nicolaides[8] extended his analysis of asymmetric missiles to include nonlinear roll orientation dependent terms and thereby introduced the concepts of "spin lock-in" and "catastrophic yaw."

The pitching and yawing motions of many rotationally symmetric finned missiles have been very well fitted by the tricyclic linear theory of slightly asymmetric missiles. These motions have been measured from ballistic range tests, free oscillation wind-tunnel tests, and full scale flight tests. They have been reduced by the classical differential corrections technique[7] or by the recently developed Wobble program.[9] In many cases the reduced frequencies and damping rates have shown a nonlinear dependence on motion amplitude.

The nonlinear analysis of the motion of slightly asymmetric missiles with symmetric nonlinear moments was briefly considered in Ref. 2. In this paper it will be expanded considerably and the special cases of zero spin rate and resonance analyzed in detail. Although most of the analysis will be applied to the case of a cubic static moment, the case of cubic damping moments[10] will be studied and the extension to higher order nonlinearities indicated. Finally the analysis will be made of the lateral center of gravity motion of a slightly asymmetric missile acted on by a cubic lift force.

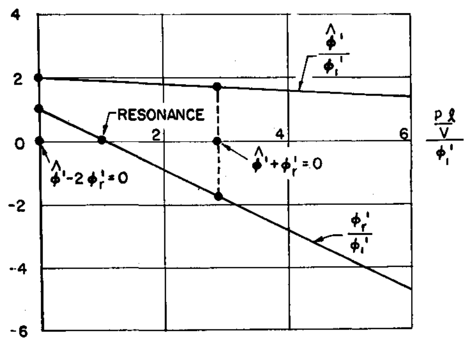

Fig. 1 Variation of $\hat{\phi}'$ and ϕ_r' with spin rate.

Nonlinear Analysis—Nonresonant Spin

If we initially limit the nonlinearities under consideration to a cubic static moment, the moment expansion for a slightly asymmetric missile assumes the form

$$C_{\tilde{m}} + iC_{\tilde{n}} = (C_{m_0} + iC_{n_0})e^{i\phi} - i(c_0 + c_2\delta^2)\xi - iC_{M_{\dot{\alpha}}}\xi' + C_{M_q}\tilde{\mu} \quad (1)$$

where

$$\delta^2 = |\xi|^2 = \xi\bar{\xi} \qquad \phi = \int_0^s (pl/V)ds_1$$

For this moment the differential equation† of motion is[11]

$$\xi'' + (H - iP)\xi' - (M_0 + M_2\delta^2)\xi = iAe^{i\phi} \quad (2)$$

where M_i, P, H, and A are defined in the Nomenclature. The solution to the linearized form of Eq. (2) with constant spin rate is the usual equation for tricyclic motion

$$\xi = K_1e^{i\phi_1} + K_2e^{i\phi_2} + K_3e^{i\phi_3} \quad (3)$$

where

$$K_j = K_{j0}e^{\lambda_j s} \qquad j = 1,2$$
$$\phi_j = \phi_{j0} + \phi_j's \qquad j = 1,2$$
$$\lambda_j = -\phi_j'H/(2\phi_j' - P)$$
$$\phi_j' = (P/2) \pm [(P^2/4) - M_0]^{1/2} \qquad \phi_3 = \phi_{30} + \phi$$
$$K_3e^{i\phi_{30}} = -iA/[(\phi')^2 - P\phi' + M_0 - i\phi'H]$$

The two modes of free oscillations are so numbered that $|\phi_1'| \geqq |\phi_2'|$. For zero spin and a statically stable missile $\phi_1' = -\phi_2'$ and the sign of ϕ_1' is arbitrarily selected to be positive. For nonzero spin the larger frequency, which is commonly called the nutational frequency by ballisticians, has the same sign as the spin. The smaller or precessional frequency has the same sign as the spin for a spin-stabilized statically unstable missile and the opposite sign for a statically stable missile.

In the case of the symmetric missile ($A = 0$) the solution curve is epicyclic ($K_3 = 0$). The nonlinear analysis for a symmetric missile assumes that the epicyclic solution is a good approximation to the actual motion and determines the average contribution of the nonlinear term $\delta^2\xi$. Since δ^2 is periodic with frequency $\hat{\phi}' = \phi_1' - \phi_2'$, this average need only be taken over one cycle of δ^2. For the slightly asymmetric missile, unfortunately two frequencies are present and the proper averaging approach requires a careful analysis. To see this we substitute Eq. (3) in Eq. (2), neglect damping terms in comparison with frequency terms, and divide by $K_1e^{i\phi_1}$

$$(\phi_1')^2 - P\phi_1' + M_0 - i[\lambda_1(2\phi_1' - P) + \phi_1'H + \phi_1''] + M_2(K_1^{-1}\delta^2\xi e^{-i\phi_1}) = -\{(\phi_2')^2 - P\phi_2' + M_0 - i[\lambda_2(2\phi_2' - P) + \phi_2'H + \phi_2'']\}(K_2/K_1)e^{-i\hat{\phi}} - \{[(\phi')^2 - P\phi' + M_0 - i\phi'H]K_3 + iAe^{-i\phi_{30}}\}K_1^{-1}e^{i\phi_r} \quad (4)$$

where $\hat{\phi} = \phi_1 - \phi_2$, and $\phi_r = \phi_1 - \phi_3$. For a statically stable, nonresonating missile $\hat{\phi}'$ and ϕ_r' are not zero and, hence, the right side of Eq. (4) is a sum of two periodic terms with zero mean values. We, therefore, average Eq. (4) over a distance which is large with respect to the wavelength of either $\hat{\phi}$ or ϕ_r

$$(\phi_1')^2 - P\phi_1' + M_0 - i[\lambda_1(2\phi_1' - P) + \phi_1'H + \phi_1''] + M_2[K_1^{-1}\delta^2\xi e^{-i\phi_1}]\text{av} = 0 \quad (5)$$

where

$$[\ \ ]\text{av} = (1/L)\int_{-L/2}^{L/2} [\ \ ]ds$$
$$L \gg \max\{2\pi/\hat{\phi}', 2\pi/\phi_r'\}$$

Similarly

$$(\phi_2')^2 - P\phi_2' + M_0 - i[\lambda_2(2\phi_2' - P) + \phi_2'H + \phi_2''] + M_2[K_2^{-1}\delta^2\xi e^{-i\phi_2}]\text{av} = 0 \quad (6)$$

$$[(\phi')^2 - P\phi' + M_0 - i\phi'H]K_3 + iAe^{-i\phi_{30}} + M_2[\delta^2\xi e^{-i\phi_3}]\text{av} = 0 \quad (7)$$

We note that Eqs. (5) and (6) determine the frequencies and damping rates with the amplitudes, K_1, K_2 and phase angles ϕ_{10}, ϕ_{20} fixed by initial conditions and Eq. (7) specifies the amplitude and phase angle of the response (K_3,ϕ_{30}).

Equations (5–7) reduce to the exact linear equation for a linear moment ($M_2 = 0$). In the case of a symmetric missile, the quantities to be averaged are functions of $\hat{\phi}$ alone and the averaging can be made over a period of $\hat{\phi}$. This yields the very successful quasi-linear relations of Refs. 1–4. The situation for the slightly asymmetric missile is much more complicated. Now

$$\delta^2 = K_1^2 + K_2^2 + K_3^2 + 2K_1K_2\cos\hat{\phi} + 2K_1K_3\cos\phi_r + 2K_2K_3\cos(\hat{\phi} - \phi_r) \quad (8)$$

$$K_1^{-1}\xi e^{-i\phi_1} = 1 + (K_2/K_1)e^{-i\hat{\phi}} + (K_3/K_1)e^{-i\phi_r} \quad (9)$$

Thus $K_1^{-1}\delta^2\xi e^{-i\phi_1}$ consists of a number of terms. In addition to constants, compelx exponential terms of frequency $\hat{\phi}'$, ϕ_r', $2\hat{\phi}'$, $2\phi_r'$, $\phi_r' - 2\hat{\phi}'$, $\hat{\phi}' - 2\phi_r'$, and $\hat{\phi}' \pm \phi_r'$ are present.

For a linear statically stable nonspinning missile $\hat{\phi}'$ is $2\phi_1'$. As spin increases $\hat{\phi}'$ slowly decreases in magnitude to ϕ_1'. ϕ_r' is ϕ_1' for zero spin and decreases in magnitude until resonant spin is reached. At this point ϕ_r' is zero. As spin is increased beyond this value ϕ_r' changes sign from the sign of ϕ_1' and increases in magnitude to very large values. These effects of spin are shown in Fig. 1.

If the vicinity of resonant spin is excluded we see that for all other values of spin six of the eight frequencies in δ^2 are the same size as either ϕ_r' or $\hat{\phi}'$. Two of these frequencies, $\hat{\phi}' - 2\phi_r'$ and $\hat{\phi}' + \phi_r'$ can become zero for certain spin rates and, therefore, terms involving them must be retained in our averaging process. To do this we introduce two angles, ϕ^* and ϕ^{**}

$$\phi^* \equiv (1/2)[\hat{\phi} - 2\phi_r] = \phi_3 - (\phi_1 + \phi_2)/2 \quad (10)$$

$$\phi^{**} \equiv (1/2)[\hat{\phi} + \phi_r] = \phi_1 - (\phi_2 + \phi_3)/2 \quad (11)$$

At zero spin $\phi_1' + \phi_2' = 0$ for a linear static moment and thus ϕ^* is a constant. Its contribution, therefore, is important for near zero spin and this special case will be considered in detail in a later section. For moderate spin rates

† The effect of lift and drag has been neglected for simplicity. These effects plus that of a linear Magnus moment can be easily added. Small geometric angles ($\delta < 0.2$) are also assumed so that certain geometric nonlinearities can be omitted.[11]

and the usual ratio of moments of inertia ($I_x/I_y \doteq 0.1$), $\phi_1' = -\phi_2'$ is a good approximation and ϕ^{**} will be constant when $\phi' \doteq 3\phi_1'$. In the vicinity of this spin rate subharmonic resonance[12] is a possibility and this will be discussed in a separate paper.

Our quasi-linear process for a slightly asymmetric missile has a special behavior in the vicinity of three spin rates—zero spin, resonance spin, and three times resonance spin. If high-order nonlinearities are considered, the number of special spin rates increases and the validity of this technique becomes quite questionable. Cubic nonlinearities may lead to reasonably accurate relations and we limit ourselves to these nonlinearities

$$[K_1^{-1}\delta^2\tilde{\xi}e^{-i\phi_1}]\text{av} = K_1^2 + 2K_2^2 + 2K_3^2 + (K_2/K_1)[K_3^2e^{2i\phi^*} + 2K_1K_3e^{-2i\phi^{**}}]\text{av} \quad (12)$$

$$[K_2^{-1}\delta^2\tilde{\xi}e^{-i\phi_2}]\text{av} = K_2^2 + 2K_1^2 + 2K_3^2 + (K_1/K_2)[K_3^2e^{2i\phi^*} + K_1K_3e^{2i\phi^{**}}]\text{av} \quad (13)$$

$$[\delta^2\tilde{\xi}e^{-i\phi_3}]\text{av} = K_3[K_3^2 + 2K_1^2 + 2K_2^2] + K_1K_2[2K_3e^{-2i\phi^*} + K_1e^{2i\phi^{**}}]\text{av} \quad (14)$$

If the spin rate is near zero, the average of exp ($2i\phi^{**}$) drops out of the previous equations and conversely if the spin rate is near $3\phi_1'$, the exp ($2i\phi^*$) terms vanish. If it is near neither of these values, both exponentials can be neglected. We will retain both terms for the remainder of this section in order that the results apply to all three possibilities.

The real parts of Eqs. (5–6) now yield relations between quasi-linear frequencies and the amplitudes of the motions for nonresonance spin rates ($\phi_r' \neq 0$)

$$\phi_j'(P - \phi_j') = M_0 + M_2\delta_{ej}^2 \qquad (j = 1,2) \quad (15)$$

where

$$\delta_{e1}^2 = K_1^2 + 2K_2^2 + 2K_3^2 + (K_2K_3^2/K_1) \times [\cos 2\phi^*]\text{av} + 2K_2K_3[\cos 2\phi^{**}]\text{av}$$

$$\delta_{e2}^2 = K_2^2 + 2K_1^2 + 2K_3^2 + (K_1K_3^2/K_2) \times [\cos 2\phi^*]\text{av} + (K_1^2K_3/K_2)[\cos 2\phi^{**}]\text{av}$$

P can be eliminated between the two Eqs. (15) with the result

$$\phi_1' \cdot \phi_2' = M_0 + M_2\delta_e^2 \quad (16)$$

where

$$\delta_e^2 = (\phi_1'\delta_{e2}^2 - \phi_2'\delta_{e1}^2)/(\phi_1' - \phi_2')$$

Eqs. (15–16) are generalized forms of the quasi-linear frequency relations for a symmetric missile and can be used in the analysis of flight data which has been fitted by the tricyclic expression of Eq. (3).

The damping rates are not explicitly affected by a cubic static moment except when the spin rate is near one of the three critical values. For near zero spin

$$\lambda_1 = -[H\phi_1' + \phi_1'' - M_2(K_2K_3^2/K_1) \times (\sin 2\phi^*)\text{av}]/(2\phi_1' - P) \quad (17)$$

$$\lambda_2 = -\{H\phi_2' + \phi_2'' - M_2(K_1K_3^2/K_2)[\sin 2\phi^*]\text{av}\}/(2\phi_2' - P) \quad (18)$$

Similar relations apply for spin rates near $3\phi_1'$. For all spin rates the damping rates are indirectly affected by M_2 through the presence of ϕ_j''.

Nonlinear Analysis—Resonance

At resonance $\phi_1' = \phi'$ and $\phi_r' = 0$. For this case, Eq. (6) for the precessional frequency is unchanged but Eqs. (5) and (7) cannot be separated

$$\{(\phi_1')^2 - P\phi_1' + M_0 - i[\lambda_1(2\phi_1' - P) + \phi_1'H + \phi_1'']\}K_1e^{i\phi_r} + [(\phi')^2 - P\phi' + M_0 - i\phi'H]K_3 + iAe^{-i\phi_{30}} + M_2[\delta^2\tilde{\xi}e^{-i\phi_3}]\text{av} = 0 \quad (19)$$

The averaged terms in Eqs. (6) and (19) need to be averaged over a period of $\hat{\phi}$ for constant ϕ_r. From Eqs. (3) and (8) we have

$$[\delta^2\tilde{\xi}K_2^{-1}e^{-i\phi_2}]\text{av} = K_2^2 + 2K_1^2 + 2K_3^2 + 4K_1K_3\cos\phi_r = (\delta_{e2}^2)_r \quad (20)$$

$$[\delta^2\tilde{\xi}e^{-i\phi_3}]\text{av} = (K_1^2 + 2K_2^2 + 2K_3^2 + K_1K_3e^{i\phi_r})K_1e^{i\phi_r} + (K_3^2 + 2K_1^2 + 2K_2^2 + K_1K_3e^{-i\phi_r})K_3 \quad (21)$$

As can be seen from Eq. (20) the precessional damping has no direct contribution from M_2 while the precessional frequency equation is

$$\phi_2'(P - \phi_2') = M_0 + M_2(\delta^2_{e2})_r \quad (22)$$

Determining the nutational frequency and damping rate as well as the response amplitude and phase angle is much more difficult. In the linear case K_3 is constant and K_1 is exponentially damped. This fact is used to require that the coefficient of K_1 in Eq. (19) be zero and so separate relations for ϕ_1', λ_1, K_3 and ϕ_{30} can be obtained. Although the nonlinear terms cannot be uniquely grouped as coefficients of K_1 and other terms, a possible grouping is given in Eq. (21). Using this grouping we have

$$\phi_1'(P - \phi_1') = M_0 + M_2(\delta^2_{e1})_r \quad (23)$$

$$(\delta_{e1}^2)_r = K_1^2 + 2K_2^2 + 2K_3^2 + K_1K_3\cos\phi_r \quad (24)$$

$$\lambda_1 = -[H\phi_1' + \phi_1'' - M_2K_1K_3\sin\phi_r]/(2\phi_1' - P) \quad (25)$$

$$[(\phi')^2 - P\phi' + M_0 + M_2(K_3^2 + 2K_1^2 + 2K_2^2 + K_1K_3e^{-i\phi_r}) - i\phi'H]K_3 + iAe^{-i\phi_{30}} = 0 \quad (26)$$

Equations (23–26) are derived for conditions which are mathematically singular and so quantitative use of the equations should be made with care. Hopefully they can give a qualitative description of the effect of a cubic moment on a slightly asymmetric missile flying at or near resonant spin rate. A different approach, which may prove to be more accurate, is given by Clare.[13]

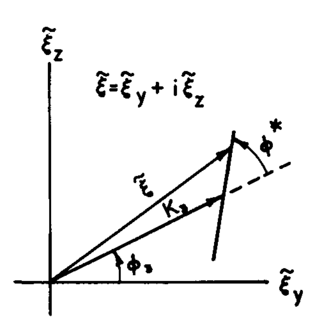

Fig. 2 Planar motion about trim for nonspinning missile.

Zero Spin Rate with Cubic Static Moment

A very simple case of nonlinear tricyclic motion which has an exact solution is the small amplitude motion of a nonspinning missile with no damping about a large trim angle

$$H = P = 0 \qquad K_j \ll K_3 \qquad j = 1,2 \quad (27)$$

For this case the quasi-linear relations for the frequencies reduce to

$$(\phi_1')^2 = -M_0\{1 + m_3[2 + (K_2/K_1)\cos 2\phi^*]\} \quad (28)$$

$$(\phi_2')^2 = -M_0\{1 + m_3[2 + (K_1/K_2)\cos 2\phi^*]\} \quad (29)$$

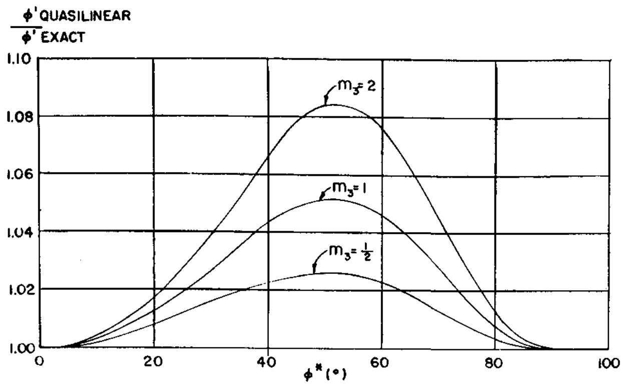

Fig. 3 Comparison of quasi-linear frequency with exact frequency.

where

$$m_3 = M_2K_3^2/M_0$$

ϕ^* will be constant when $\phi_1' = -\phi_2'$ and this, according to Eqs. (28–29), will happen when the motion about the trim angle is planar motion, i.e., $K_1 = K_2$.

The damping rates are given by Eqs. (17–18). If we differentiate Eqs. (28–29) to obtain ϕ_j'' and make use of assumption (27), we can obtain the damping rates as well as the frequencies for $K_1 = K_2$

$$\lambda_j = M_2K_3^2 \sin 2\phi^*/2\phi_j'[1 + M_2K_3^2 \cos 2\phi^*/2(\phi_1')^2] \quad (30)$$

$$\phi_1' = -\phi_2' = \{-M_0[1 + m_3(2 + \cos 2\phi^*)]\}^{1/2} \quad (31)$$

Since $\phi_1' = -\phi_2'$, Eq. (30) states that $\lambda_1 = -\lambda_2$ and, therefore, planar motion will remain planar when ϕ^* is 0 or $\pi/2$. It can easily be shown that ϕ^* is the angle between the planar motion about the trim angle and the plane of the trim angle of attack. Thus we see that this motion will remain planar only when it is in the plane of the trim angle or perpendicular to that plane (Fig. 2).

Under assumption (27) the differential Eq. (2) becomes linear in $\hat{\alpha}$ and $\hat{\beta}$ where

$$\xi = [\hat{\beta} + i\hat{\alpha} + K_3]e^{i\phi_{30}} \quad (32)$$

Substituting Eq. (32) in Eq. (2) and solving for initially planar motion ($\hat{\alpha}_0 = \hat{\beta}_0 = 0$) we have

$$K_3(M_0 + M_2K_3^2)e^{i\phi_{30}} = -iA \quad (33)$$

$$\hat{\alpha} = B_\alpha \sin\omega_\alpha s \qquad \hat{\beta} = B_\beta \sin\omega_\beta s \quad (34)$$

$$\omega_\alpha^2 = -M_0(1 + m_3) \qquad \omega_\beta^2 = -M_0(1 + 3m_3) \quad (35)$$

The orientation of the initial planar motion can be computed by the relation

$$\tan\phi^* = \hat{\alpha}_0'/\hat{\beta}_0' = \omega_\alpha B_\alpha/\omega_\beta B_\beta \quad (36)$$

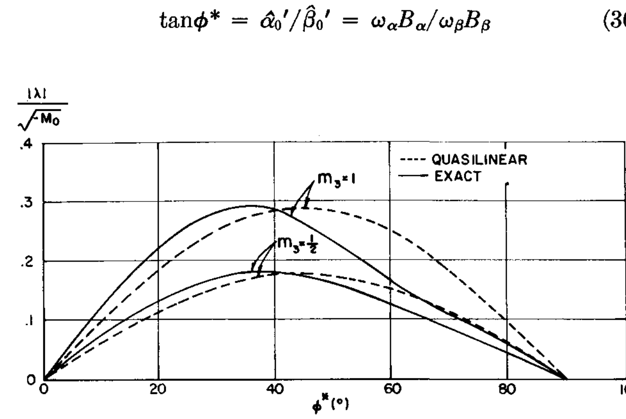

Fig. 4 Comparison of quasi-linear damping rate with exact damping rate.

Table 1 Effective damping rates for cubic nonlinear damping

$\phi' = 0$	
$\lambda_1^* =$	$-(\frac{1}{2})[H_0 + H_2(K_1^2 + K_3^2) - M_2^*(2K_2^2 + K_3^2 + (K_3^2K_2/K_1)\cos 2\phi^*) - M_2(\phi_1')^{-1}K_2K_3^2K_1^{-1}\sin 2\phi^*]$
$\lambda_2^* =$	$-(\frac{1}{2})[H_0 + H_2(K_2^2 + K_3^2) - M_2^*[2K_1^2 + K_3^2 + (K_3^2K_1/K_2)\cos 2\phi^*] - M_2(\phi_2')^{-1}K_1K_3^2K_2^{-1}\sin 2\phi^*]$
	ϕ' not near 0, ϕ_1', $3\phi_1'$ $(\phi_1' \doteq -\phi_2')^a$
$\lambda_1^* =$	$-(\frac{1}{2})\{H_0 + H_2[K_1^2 + (1 + \phi'/\phi_1')K_3^2] - M_2^*[2K_2^2 + (1 - \phi'/\phi_1')K_3^2]\}$
$\lambda_2^* =$	$-(\frac{1}{2})\{H_0 + H_2[K_2^2 + (1 + \phi'/\phi_2')K_3^2] - M_2^*[2K_1^2 + (1 - \phi'/\phi_2')K_3^2]\}$

[a] These relations can and should be recalculated if this assumption is invalid.

We note that the quasi-linear relation for frequency [Eq. (31)] reduces to the exact frequencies ω_α or ω_β when ϕ^* is 0 or $\pi/2$. Indeed ω_β^2 is the derivative of the moment curve at the value of the trim angle which is the well-known linear relation for motion about trim in the plane of the trim angle. ω_α^2, which is the frequency of small amplitude motion perpendicular to the plane of the trim angle, depends on the *moment* at the trim angle not its derivative since for this motion the magnitude of the moment does not change for small angles.

At intermediate values of ϕ^* the exact solution is the vector sum of two perpendicular motions of different frequency and is quite different from an epicyclic motion. We can determine how well the quasi-linear relation predicts the first maximum of δ by finding s for δ_{max} from Eq. (34), computing an approximate frequency from it, and comparing with the quasi-linear value. This is done in Fig. 3 for three values of m_3; thus the quasi-linear relation is accurate to 5%.

Approximate damping rates can be determined from the first maximum and the first minimum of δ

$$\delta_{max} = K_1 + K_2 = K_0(e^{\lambda s_1} + e^{-\lambda s_1}) = 2K_0 \cosh\lambda s_1 \quad (37)$$

Similarly,

$$\delta_{min} = 2K_0 \sinh\lambda s_2 \qquad \lambda = |\lambda_j| \quad (38)$$

λ was calculated by use of Eqs. (37–38) for the exact solution and is compared with the quasi-linear λ in Fig. 4. The agreement for rather large values of m_3 is surprisingly good.

Cubic Damping Moments

The only nonlinear damping moment which has been reliably determined from ballistic range data is a cubic Magnus moment. Since a tricyclic motion is usually present for relatively low spin rates, a Magnus moment is usually neglected. The simplest nonlinear damping moment for a slowly spinning missile has the form[10]

$$C_{\tilde{m}} + iC_{\tilde{n}} = -i\{[c_0 + c_2\delta^2 + c_2^*(\delta^2)']\xi + [d_0 + d_2\delta^2]\xi'\} + (C_{m_0} + iC_{n_0})e^{i\phi} \quad (39)$$

The corresponding differential equation takes on the form

$$\xi'' + (H_0 + H_2\delta^2 - iP)\xi' - [M_0 + M_2\delta^2 + M_2^*(\delta^2)']\xi = iAe^{i\phi} \quad (40)$$

We can usually assume a moment of inertia ratio such that $P \ll \phi_1'$ and, therefore, $\phi_1' \doteq -\phi_2'$. The damping rates can then be computed to be

$$\lambda_j = \lambda_j^* - \phi_j''/2\phi_j' \quad (41)$$

where λ_j^* are given in Table 1. The algebraic work is quite similar to that for a cubic static moment and will not be repeated here.

Swerving Motion with Cubic Lift

The swerving motion of a missile is the motion induced by an aerodynamic force perpendicular to its average flight path. If we neglect gravity and Coriolis effects the equation of motion for a symmetrical lift force plus a missile attached asymmetric normal force is

$$(x_2'' + ix_3'')/l = f\tilde{\xi} - ge^{i\phi} \qquad (42)$$

where

$$f = (\rho Sl/2m)C_{L_\alpha} \qquad g = (\rho Sl/2m)(C_{Y_0} + iC_{Z_0})$$

For constant f, g, ϕ', K_1 and K_2, this equation can be integrated to yield

$$(x_2 + ix_3)/l = B_0 + B_1 s + fI_1 - (g - fK_3 e^{i\phi_{30}})I_2 \qquad (43)$$

where

$$I_1 = -K_1 e^{i\phi_1}/(\phi_1')^2 - K_2 e^{i\phi_2}/(\phi_2')^2$$

$$I_2 = \int_0^s \int_0^{s_2} e^{i\phi} ds_1 ds_2 = -(e^{i\phi} - 1 - i\phi' s)/(\phi')^2$$

Note that all terms involving the spin rate are grouped with I_2. This allows us to handle the singular case of zero spin in a convenient way. If we assume a cubic lift force,

$$f = f_0 + f_2\delta^2 \qquad (44)$$

Ballistic range data for swerving motion can still be fitted by Eq. (43) even when the lift force is cubic. The coefficient of I_1 is, however, an average "range" value of f which is a linear combination of f_0 and f_2

$$f_R = f_0 + f_2\delta^2_{es} \qquad (45)$$

We now would like to find a relation for this effective value of δ^2 for swerve.

The nonlinear term $\delta^2\tilde{\xi}$ contains twelve combinations of the three basic frequencies. We group the two terms which have zero frequency for zero spin rate with the five terms that always have the spin frequency so that

$$\int_0^s \int_0^{s_2} \delta^2\tilde{\xi} ds_1 ds_2 = B_{0c} + B_{1c}s + I_{1c} + I_{2c} \qquad (46)$$

where I_{1c} = sum of periodic terms which are periodic for zero spin.

$$I_{2c} = K_3 \int_0^s \int_0^{s_2} [2K_1^2 + 2K_2^2 + K_3^2 + 2K_1K_2 e^{-i2\phi^*}]e^{i\phi_3} ds_1 ds_2$$

We now want to fit I_{1c} to $I_1\delta_{es}^2$ by least squares

$$\int_0^L |I_1\delta_{es}^2 - I_{1c}|^2 ds \text{ is a minimum}$$

Differentiating and solving for δ_{es}^2 we have

$$\delta_{es}^2 = (1/L)\int_0^L (I_1\bar{I}_{1c} + \bar{I}_1 I_{1c})ds \Big/ (2/L)\int_0^L I_1\bar{I}_1 ds \qquad (47)$$

but

$$(2/L)\int_0^L I_1\bar{I}_1 ds = 2[K_1^2/(\phi_1')^4 + K_2^2/(\phi_2')^4] \qquad (48)$$

Since I_1 contains ϕ_1' and ϕ_2', the only terms in I_{1c} which can make contributions to the numerator of Eq. (47) are those with frequencies close to ϕ_1' or ϕ_2'. Assuming that ϕ' is not near ϕ_1' or $3\phi_1'$, we can now compute this numerator

$$(1/L)\int_0^L (I_1\bar{I}_{1c} + I_{1c}\bar{I}_1)ds = 2(K_1^2(K_1^2 + 2K_2^2 + 2K_3^2)/(\phi_1')^4 + K_2^2(K_2^2 + 2K_1^2 + 2K_3^2)/(\phi_2')^4 + K_1K_2K_3^2[\cos 2\phi^*]\text{av}\{[\phi_1'(2\phi' - \phi_2')]^{-2} + [\phi_2'(2\phi' - \phi_1')]^{-2}\}) \qquad (49)$$

For near zero spin $\phi_1' \doteq -\phi_2'$ and $\delta_{es}^2 =$

$$\{K_1^4 + 4K_1^2K_2^2 + K_2^4 + 2K_3^2[K_1^2 + K_2^2 + K_1K_2(\cos 2\phi^*)\text{av}]\}/(K_1^2 + K_2^2) \qquad (50)$$

For nonzero spin $\delta_{es}^2 =$

$$[(\phi_2')^4K_1^2(K_1^2 + 2K_2^2 + 2K_3^2) + (\phi_1')^4K_2^2(K_2^2 + 2K_1^2 + 2K_3^2)]/[(\phi_2')^4K_1^2 + (\phi_1')^4K_2^2] \qquad (51)$$

Summary

Quasi-linear relations for frequencies, damping rates, and amplitude of swerving motions have been derived for slight asymmetric missiles with cubic static and damping moments and cubic lift forces.

These relations take on special values for spin rates of 0, ϕ_1', $3\phi_1'$.

The quasi-linear relations have been shown to be quite accurate for a nonspinning missile performing small amplitude motion about a large trim angle.

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